University College London
DEPARTMENT OF MATHEMATICS
Mid-Sessional Examinations 2009
Mathematics 1201
Monday 12 January 2009 11.30 – 1.30 or 1.15 – 3.15

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination*

1) (i) Replace the negation of the following formula by an equivalent formula which does not involve $\neg$, $\wedge$ or $\vee$;

$$(\forall x)(\exists y)\,(P(x) \vee \neg Q(y)) \bigwedge (\forall y)(\exists x)\,(P(y) \wedge \neg Q(x)).$$

(ii) Let $f : A \to B$ be a mapping between sets A, B. Explain what is meant by saying that (a) f is injective ; (b) f is invertible. Show that an invertible mapping is injective.
In each case below decide, giving your explanation, whether the given mapping is a) injective b) surjective ;
i) $f : \mathbf{Z} \to \mathbf{Z}$; $f(x) = x^3 + x$;
ii) $g : \mathbf{R} \to \mathbf{R}$; $g(x) = x^3 - x$.

2) Let $\epsilon(r,s)$ be the basic $m \times m$ matrix given by $\epsilon(r,s)_{ij} = \delta_{ri}\delta_{sj}$ where 'δ' denotes the Kronecker delta. Explain with proof how to calculate the product $\epsilon(r,s)\epsilon(u,t)$.
Describe in detail the elementary $m \times m$ matrices
(i) $E(r,s;\lambda)$ $(r \neq s)$; (ii) $\Delta(r,\lambda)$ $(\lambda \neq 0)$; (iii) $P(r,s)$ $(r \neq s)$
in terms of the basic matrices $\epsilon(r,s)$.
Find the inverse A^{-1} of the matrix A below. Hence express both A^{-1} and A as products of elementary matrices.

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 3 & 2 \\ 3 & 2 & 0 \end{pmatrix}.$$

PLEASE TURN OVER

3) Let V, W be vector spaces over a field $\mathbf{F}$ and let $T : V \to W$ be a mapping; explain what is meant by saying that T is *linear*.
When T is linear, explain what is meant by
(a) the kernel, $\mathrm{Ker}(T)$ and
(b) the image, $\mathrm{Im}(T)$.
State and prove a relationship which holds between $\dim \mathrm{Ker}(T)$ and $\dim \mathrm{Im}(T)$.

Let $T_A : \mathbf{Q}^6 \to \mathbf{Q}^4$ be the linear mapping $T_A(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & 3 & 0 & 0 & 2 & 1 \\ 1 & 3 & -1 & 0 & 2 & -1 \\ 1 & 2 & 3 & 3 & 5 & 4 \\ 1 & 0 & 4 & 3 & 5 & 6 \end{pmatrix}.$$

Find (i) $\dim \mathrm{Ker}(T_A)$; (ii) a basis for $\mathrm{Ker}(T_A)$; (iii) a basis for $\mathrm{Im}(T_A)$.

4) Let $T : U \to V$ be a linear map between vector spaces U , V, and let $\mathcal{E} = (e_i)_{1\le i\le m}$ be a basis for U and $\Phi = (\varphi_j)_{1\le j\le n}$ be a basis for V. Explain what is meant by the matrix $m(T)_{\mathcal{E}}^{\Phi}$ of T taken with respect to $\mathcal{E}$ (on the left) and Φ (on the right) and prove that if $S : V \to W$ is also a linear map and $\Psi = (\psi_k)_{1\le k\le p}$ is a basis for W then

$$m(S \circ T)_{\mathcal{E}}^{\Psi} = m(S)_{\Phi}^{\Psi} m(T)_{\mathcal{E}}^{\Phi}.$$

Hence derive a relationship between $m(\mathrm{Id})_{\Phi}^{\mathcal{E}}$ and $m(\mathrm{Id})_{\mathcal{E}}^{\Phi}$ when $U = V = W$.

Let $\mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$; $\Phi = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$

be bases for $\mathbf{Q}^3$ and let $T : \mathbf{Q}^3 \to \mathbf{Q}^3$ be a linear mapping. Express $m(T)_{\mathcal{E}}^{\mathcal{E}}$ in terms of $m(T)_{\Phi}^{\Phi}$ and $m(\mathrm{Id})_{\Phi}^{\mathcal{E}}$, and hence find $m(T)_{\mathcal{E}}^{\mathcal{E}}$ when

$$m(T)_{\Phi}^{\Phi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

CONTINUED

5) Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a subset of a vector space V; explain what is meant by saying that the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent.

In each case below, decide with justification whether the given vectors are linearly independent. If they are not, give an explicit dependence relation between them.

(a) $$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 3 \\ -1 \\ 2 \end{pmatrix};$$

(b) $$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 3 \\ 0 \\ 2 \end{pmatrix}.$$

Explain what is meant by a *spanning set* for a vector space V. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a spanning set for V, and suppose that $\mathbf{u} \in V$ can be expressed as a linear combination of the form

$$\mathbf{u} = \sum_{r=1}^{n} \lambda_r \mathbf{v}_r$$

with $\lambda_1 \neq 0$. Show that $\{\mathbf{u}, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is also a spanning set for V.

State and prove the Exchange Lemma.

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6) (i) Let σ be a permutation of the set $\{1, \ldots, n\}$; explain what is meant by saying that (a) σ is a transposition; (b) σ is an adjacent transposition. Show that any transposition can be written as a product of adjacent transpositions.

(ii) Write $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 14 & 6 & 12 & 11 & 10 & 4 & 1 & 5 & 13 & 8 & 2 & 7 & 9 & 3 \end{pmatrix}$

as a product of disjoint cycles and hence compute $\text{sign}(\sigma)$ and $\text{ord}(\sigma)$.

(iii) Let $\mathcal{P}_7(\mathbf{R})$ be the vector space of polynomials of degree ≤ 7 over the field $\mathbf{R}$ and let $D : \mathcal{P}_7(\mathbf{R}) \to \mathcal{P}_7(\mathbf{R})$ be the linear map given by differentiation. Write down the least positive integer n for which $D^{2n} = 0$ on $\mathcal{P}_7(\mathbf{R})$.

By factorising $D^{2n} - \mathrm{I}$ show that the mapping

$$D^4 + \mathrm{I} : \mathcal{P}_7(\mathbf{R}) \to \mathcal{P}_7(\mathbf{R})$$

is invertible, and write down

(i) an expression for its inverse in terms of D, and

(ii) the unique solution $\alpha \in \mathcal{P}_7(\mathbf{R})$ to the differential equation

$$\frac{d^4\alpha}{dx^4} + \alpha \quad = \quad x^6 - x^7.$$

END OF PAPER